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Necessary and sufficient range-dimension conditions for bipartite quantum correlations

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Abstract
As is well known, every mixed or pure state of a bipartite quantum system is given by a statistical operator, which determines, in terms of its two reduced statistical operators, the subsystem states. Necessary and sufficient conditions for the existence of a composite-system state, and, separately, for the possibility of its being correlated or uncorrelated in terms of the range dimensions of the three mentioned statistical operators are derived. As a corollary, it is shown that it cannot happen that two of the mentioned dimensions are finite and the third is infinite.

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It is assumed throughout that one can speak of quantum correlations in a composite (1 + 2) quantum system if and only if a correlated composite-system statistical operator ρ_3 is given. (The index 3 instead of 12 is used for reasons that will become clear below in condition (1).) Let, further,

$$\rho_1 \equiv \mathrm{Tr}_2\, \rho_3 \qquad \rho_2 \equiv \mathrm{Tr}_1\, \rho_3$$

be the reduced statistical operators (physically, the states of the subsystems), where 'Tr_2' and 'Tr_1' denote the respective partial traces. Let the dimensions of the ranges $\mathcal{R}(\rho_i)$, i.e., the *range dimensions*, be denoted by d_i, $i = 1, 2, 3$.

Physically, a statistical operator ρ_3 is a general, i.e., a pure or mixed, state of a composite system. It is *uncorrelated* if $\rho_3 = \rho_1 \otimes \rho_2$, and it is *correlated* otherwise.

The theorem to be proved in this paper works with three range-dimension conditions:

The cyclic inequality conditions:

$$d_1 \leqslant d_2 d_3 \qquad d_2 \leqslant d_3 d_1 \qquad d_3 \leqslant d_1 d_2 \tag{1}$$

the common lower bound condition:

$$2 \leqslant d_1, d_2 \tag{2}$$

and, finally,
the product condition:

$$d_3 = d_1 d_2. \tag{3}$$

It is easily seen that (3) implies (1) and obviously the former does not follow from the latter, i.e., (3) is a *stronger requirement* than (1).

The *theorem* on *range-dimension conditions* for bipartite states goes as follows:

(i) For every (correlated or uncorrelated) state ρ_3 conditions (1) are valid; and vice versa, for every three natural numbers d_1, d_2, d_3 satisfying conditions (1) there exists at least one state ρ_3 implying them as its range dimensions.
(ii) One can have a *correlated* composite-system state ρ_3 if and only if, in addition to the cyclic inequality conditions (1), also the common lower bound condition (2) is valid.
(iii) For every *uncorrelated* state ρ_3 condition (3) is valid; and vice versa, if three natural numbers d_1, d_2, d_3 satisfy condition (3), then there exists at least one uncorrelated state ρ_3 for which these numbers are the range dimensions.

At first sight one may be puzzled because 'correlated' and 'uncorrelated' are mutually exclusive concepts for states, and the corresponding claimed conditions do not exclude each other: both conditions (ii) and (iii) can be simultaneously valid.

The answer, of course, lies in the fact that in the mentioned case both correlated and uncorrelated states exist. Namely, the 'sufficiency' in the condition does not claim that the condition necessarily implies correlations or lack of them respectively; it only implies the existence of a correlated (or an uncorrelated) state with the given range dimensions.

It should be pointed out that none of the three dimensions is assumed to be finite. Conditions (1) will be seen to follow from a remarkable fact:

$$d_3 = 1 \quad \Rightarrow \quad d_2 = d_1 \tag{4}$$

which is known [1], but, perhaps, not well known. It is easy to see that condition (4) follows from (1), but the latter has a wider scope than the former: it covers all states, not only the pure ones ($d_3 = 1$).

It is a *corollary* of conditions (1) that *one cannot have precisely one of the three dimensions infinite*. If, e.g., d_1 were infinite, and d_2 and d_3 were finite, this would contradict the first inequality in (1). The symmetrical arguments hold for the other two cases. The conditions obviously allow all three of the dimensions or any two or none to be infinite.

The theorem is a modest contribution to the study of quantum correlations, and the latter are important for quantum information theory, as well as for quantum communication and quantum computation theories [2].

The rest of this paper is devoted to *a proof of the theorem*. We begin by proving claim (i).

To prove the necessity of the first condition in (1), we assume that an arbitrary composite-system statistical operator ρ_3 is given. Every such operator has a purely discrete (finite or infinite) spectrum ([3], theorems VI.16 and VI.21). Hence, we can write it in a spectral form:

$$\rho_3 = \sum_{n=1}^{d_3} r_n |\Psi^{(n)}\rangle_3 \langle\Psi^{(n)}|_3. \tag{5}$$

On account of (4), one has

$$d_1^{(n)} = d_2^{(n)} \qquad n = 1, 2, \ldots, d_3 \tag{6}$$

where the symbols $d_i^{(n)}$ denote the respective dimensions of the ranges $\mathcal{R}\big(\rho_i^{(n)}\big)$, and $\rho_i^{(n)}$ are the reduced statistical operators of the pure characteristic states $|\Psi^{(n)}\rangle_3$ in (5), $i = 1, 2; \quad n = 1, 2, \ldots, d_3$.

Taking the partial trace over subsystem 2 in (5), one obtains

$$\rho_1 = \sum_{n=1}^{d_3} r_n \rho_1^{(n)}. \tag{7}$$

We replace each $\rho_1^{(n)}$ in (7) by a spectral decomposition into pure states with positive characteristic values, i.e., we write

$$\rho_1 = \sum_{n=1}^{d_3} r_n \sum_{j=1}^{d_1^{(n)}} r_j^{(n)} \left|\phi_j^{(n)}\right\rangle_1 \left\langle\phi_j^{(n)}\right|_1. \tag{8}$$

Let us recall the known fact that, in general, the state vectors corresponding to the pure states of which the state is a mixture span (as linear combinations and limiting points) the topological closure $\bar{\mathcal{R}}(\rho)$ of the range of the statistical operator ρ that corresponds to the mixture. Hence, one can conclude that

$$\bar{\mathcal{R}}(\rho_1) = \sum_{n=1}^{d_3} \bar{\mathcal{R}}\left(\rho_1^{(n)}\right) \tag{9}$$

is valid. (The sum in (9) is an ordinary sum of subspaces, i.e., the LHS is the linear and topological span of the union of the RHS subspaces. The terms need not be linearly independent, let alone orthogonal.)

As to the dimensions, (9) evidently implies

$$d_1^{(n)} \leqslant d_1 \leqslant \sum_{n'=1}^{d_3} d_1^{(n')} \qquad n = 1, 2, \ldots, d_3. \tag{10}$$

(The second equality is achieved if the sum in (9) is a direct one or, in particular, an orthogonal one.) Naturally, also the symmetrical inequalities hold true (and they are proved by the symmetrical argument):

$$d_2^{(n)} \leqslant d_2 \leqslant \sum_{n'=1}^{d_3} d_2^{(n')} \qquad n = 1, 2, \ldots, d_3. \tag{11}$$

Substituting (6) in the second inequality in (10), one obtains

$$d_1 \leqslant \sum_{n=1}^{d_3} d_2^{(n)}.$$

Utilizing the first inequality in (11), one further has

$$d_1 \leqslant \sum_{n=1}^{d_3} d_2 = d_2 d_3.$$

The second inequality in (1), i.e., $d_2 \leqslant d_3 d_1$, is proved symmetrically. The last relation in (1), i.e., $d_3 \leqslant d_1 d_2$, is known (see, e.g., [4], relation (11) therein).

To prove sufficiency of the three inequalities in (1), we assume first that d_3 is the dominant quantity, i.e., that we have

$$d_2 \leqslant d_1 \leqslant d_3 \tag{12}$$

and we give a construction of a statistical operator ρ_3 having the given dimensions. (Within the case of dominance of d_3, the other possibility, namely $d_1 \leqslant d_2$, is handled symmetrically.)

Further, we take an orthonormal (ON) basis $\{|\phi^{(i)}\rangle_1 : i = 1, 2, \ldots, d_1\}$ spanning the range $\mathcal{R}(\rho_1)$, and an ON basis $\{|\chi^{(j)}\rangle_2 : j = 1, 2, \ldots, d_2\}$ spanning the range $\mathcal{R}(\rho_2)$. (If a range is infinite dimensional, then it is understood that an ON basis spans the topological closure of the range.) Proceeding further, we make the direct products: $\bar{\rho}_3^{(i,j)} \equiv |\phi^{(i)}\rangle_1\langle\phi^{(i)}|_1 \otimes |\chi^{(j)}\rangle_2\langle\chi^{(j)}|_2$, for all pairs (i, j).

Next, we form the sequence

$$\rho_3^{(n=1)} \equiv \bar{\rho}_3^{(1,1)}, \quad \rho_3^{(n=2)} \equiv \bar{\rho}_3^{(2,2)}, \ldots, \quad \rho_3^{(n=d_2)} \equiv \bar{\rho}_3^{(d_2,d_2)}$$

$$\rho_3^{(n=d_2+1)} \equiv \bar{\rho}_3^{(d_2+1,1)}, \quad \rho_3^{(n=d_2+2)} \equiv \bar{\rho}_3^{(d_2+2,1)}, \ldots, \quad \rho_3^{(n=d_1)} \equiv \bar{\rho}_3^{(d_1,1)}$$

and join to it any $(d_3 - d_1)$ states $\bar{\rho}_3^{(i,j)}$ from the rest as

$$\rho_3^{(n=d_1+1)}, \ldots, \quad \rho_3^{(n=d_3)}.$$

Since $d_3 \leqslant d_1 d_2$ (the third inequality in (1)), there is a sufficient number of $\bar{\rho}_3^{(i,j)}$ states for this. If $d_2 = d_1$, then the second row is, of course, omitted. Analogously, if $d_1 = d_3$, the last subset of states is omitted. Finally, we take a decomposition of 1 into d_3 positive numbers w_n: $1 = \sum_{n=1}^{d_3} w_n$, and the constructed ρ_3 is by definition the mixture

$$\rho_3 \equiv \sum_{n=1}^{d_3} w_n \rho_3^{(n)}.$$

It is easy to see that this state, which is a mixture of orthogonal pure states, has the dimensions $d_i, i = 1, 2, 3$, given at the beginning of our sufficiency proof for (1).

To proceed with our proof of sufficiency of the three inequalities in condition (1), for the existence of a composite-system statistical operator ρ_3 with the given dimensions, we now assume that d_3 is not the dominant quantity, i.e., that we have

$$d_2, d_3 \leqslant d_1. \tag{13}$$

(Again, the subcase $d_1, d_3 \leqslant d_2$ is treated symmetrically.) Further, we give a construction of a statistical operator ρ_3 having the given dimensions.

We construct ρ_3 in the spectral form

$$\rho_3 = \sum_{n=1}^{d_3} r_n |\Psi^{(n)}\rangle_3\langle\Psi^{(n)}|_3.$$

The characteristic values $\{r_n : n = 1, 2, \ldots, d_3\}$ are arbitrary fixed positive numbers such that $\sum_{n=1}^{d_3} r_n = 1$.

To construct the characteristic vectors $|\Psi^{(n)}\rangle_3$, we introduce an ON basis $\{|i\rangle_1 : i = 1, 2, \ldots, d_1\}$ spanning $\mathcal{R}(\rho_1)$, and another ON basis $\{|j\rangle_2 : j = 1, 2, \ldots, d_2\}$ spanning $\mathcal{R}(\rho_2)$. We break up the former basis into d_3 disjoint sets of basis vectors, i.e., into subbases, each containing at most d_2 vectors. (This is possible because of the first inequality in (1).) We enumerate the subbases by $n = 1, 2, \ldots, d_3$. Let D_n be the number of basis vectors in the nth subbasis. Within this subbasis we enumerate the vectors by a subset of the indices of the chosen ON basis in $\mathcal{R}(\rho_2)$ as follows:

$$j = \left(\sum_{n'=1}^{n-1} D_{n'}\right) + 1, \quad \left(\sum_{n'=1}^{n-1} D_{n'}\right) + 2, \ldots, \quad \left(\sum_{n'=1}^{n-1} D_{n'}\right) + D_n.$$

(In the first subbasis the sums in parentheses are, of course, omitted.) When j reaches the value d_2, we count its values further cyclically: $j = d_2 + 1 \equiv 1, \quad j = d_2 + 2 \equiv 2$, etc.

Then we construct

$$|\Psi^{(n)}\rangle_3 \equiv \sum_j \alpha_j |j\rangle_1 \otimes |j\rangle_2 \qquad n = 1, 2, \ldots, d_3$$

where the α_j are arbitrary nonzero complex numbers such that $\sum_j |\alpha_j|^2 = 1$ for each value of n independently, and 'j' enumerates the vectors $|i\rangle_1$ within the nth subbasis.

Obviously, on account of the disjointness of the mentioned subbases,

$$\langle \Psi^{(n)} || \Psi^{(n')} \rangle = \delta_{n,n'}.$$

It is easily seen that the range dimensions of the constructed ρ_3 are precisely the initially given quantities d_1, d_2, d_3. This completes the proof of claim (i).

To prove claim (ii), we first prove the *necessity* of the common lower bound condition (2) *ab contrario*.

Lemma. *If $d_1 = 1$ or $d_2 = 1$ or both, then the corresponding state ρ_3 is necessarily uncorrelated, i.e., $\rho_3 = \rho_1 \otimes \rho_2$.*

Proof. We assume $d_1 = 1$ (with no assumption on d_2). Let $|a\rangle_1$ be a state vector spanning $\mathcal{R}(\rho_1)$, and let $\{|j\rangle_2 : j = 1, 2, \ldots, d_2\}$ be an ON basis spanning $\mathcal{R}(\rho_2)$. Since $\mathcal{R}(\rho_3) \subseteq (\mathcal{R}(\rho_1) \otimes \mathcal{R}(\rho_2))$ (cf again [4], relation (11) therein), we can expand ρ_3 in the dyadic operator basis:

$$\rho_3 = \sum_j \sum_{j'} r_{jj'} |a\rangle_1 \langle a|_1 \otimes |j\rangle_2 \langle j'|_2$$

$$= |a\rangle_1 \langle a|_1 \otimes \sum_j \sum_{j'} r_{jj'} |j\rangle_2 \langle j'|_2.$$

Taking the partial traces, one infers that $\rho_1 = |a\rangle_1 \langle a|_1$, and that $\rho_2 = \sum_j \sum_{j'} r_{jj'} |j\rangle_2 \langle j'|_2$, and, finally, that $\rho_3 = \rho_1 \otimes \rho_2$ as claimed. The case $d_2 = 1$ and $d_1 \neq 1$ is proved symmetrically.

To prove *sufficiency* of condition (2) in conjunction with (1) for claim (ii), one should note that both constructions in the above proof of claim (i) easily give a correlated state in this case.

To *prove claim* (iii), we begin by *necessity*. Let $\rho_3 = \rho_1 \otimes \rho_2$, and let $\{|i\rangle_1 : i = 1, 2, \ldots, d_1\}$ and $\{|j\rangle_2 : j = 1, 2, \ldots, d_2\}$ be characteristic subbases of ρ_1 and ρ_2, respectively, spanning the respective ranges. Then

$$\rho_3 = \sum_{i=1}^{d_1} \sum_{j=1}^{d_2} r_i r_j |i\rangle_1 \langle i|_1 \otimes |j\rangle_2 \langle j|_2$$

where the r_i and r_j are the corresponding characteristic values of ρ_1 and ρ_2, respectively. From this characteristic decomposition of ρ_3 with $d_1 d_2$ terms one infers that the product condition (3) is valid.

To prove *sufficiency*, we construct any ρ_1 and ρ_2 with the given range dimensions d_1 and d_2, respectively, and multiply them: $\rho_3 \equiv \rho_1 \otimes \rho_2$.

□

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